Well-posedness of a singular balance law

Boris Andreianov Nicolas Seguin

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Abstract

We define entropy weak solutions and establish well-posedness for the Cauchy problem for the formal equation

$$\partial_t u(t,x) + \partial_x \frac{u^2}{2}(t,x) = -\lambda\, u(t,x)\, \delta_0(x),$$

which can be seen as two Burgers equations coupled in a non-conservative way through the interface located at $x = 0$. This problem appears as an important auxiliary step in the theoretical and numerical study of the one-dimensional particle-in-fluid model developed by Lagoutière, Seguin and Takahashi [LST08].

The interpretation of the non-conservative product "$u(t,x)\,\delta_0(x)$" follows the analysis of [LST08]; we can describe the associated interface coupling in terms of one-sided traces on the interface. Well-posedness is established using the tools of the theory of conservation laws with discontinuous flux ([AKR11]).

For proving existence and for practical computation of solutions, we construct a finite volume scheme, which turns out to be a well-balanced scheme and which allows a simple and efficient treatment of the interface coupling. Numerical illustrations are given.

Key words. Burgers equation, Fluid-particle interaction, Non-conservative coupling, Singular source term, Well-posedness, Interface traces, Adapted entropies, Finite volume scheme, Well-balanced scheme, Convergence

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Contents

1 Introduction

1.1 The problem

In this paper we focus on the following Cauchy problem:

$$\begin{cases} \partial_t u(t,x) + \partial_x (u^2/2)(t,x) = -\lambda\ u(t,x)\ \delta_0(x), & t>0,\ x\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{1}$$

where $\lambda > 0$ and $u_0 \in L^\infty(\mathbb{R})$. A succinct account on problem (1) was presented in [ALST10]; here we provide some further results and give detailed proofs.

1.2 Relation to a fluid-particle interaction model

Problem (1) derives from the model, studied in [LST08], for the interaction of a particle with a fluid in space dimension one. The fluid is modelled by the Burgers equation, the size of the particle is neglected, and the interaction takes place *via* a drag force concentrated at the particle location:

$$\partial_t u + \partial_x (u^2/2) = \lambda\ D(h'(t) - u)\ \delta_0(x - h(t)), \tag{2}$$

$$m h''(t) = \lambda\ D(u(t,h(t)) - h'(t)). \tag{3}$$

Here the two unknowns are u, the velocity of the fluid, and h, the position of the solid particle. The drag coefficient λ and the mass m of the particle are positive parameters. The function D which intervenes in the drag force is assumed linear ($D(v) = v$). Quadratic drag force ($D(v) = v|v|$) also appears in modelling; it leads to a simpler study. Problem (1) with the quadratic drag force (*i.e.*, the case where $-\lambda u|u|\delta_0$ replaces the right-hand side of (1)) is briefly described in Appendix B.

A natural approach for the theoretical study of the system (2)–(3) is the fixed-point algorithm with decoupling of the two equations; the numerical counterpart is a splitting scheme were the states u and h are updated alternatively. Problem (1) appears as a step of the fixed-point argument aimed at existence of solutions to (2)–(3) (see [ALST10, ALST]). Indeed, our problem (1) corresponds to equation (2) driven by the particle which is assumed to be at rest ($h(t) \equiv 0$); moreover, if the particle moves along the straight line $h(t) = Vt$, then (2) also reduces to (1) by the simultaneous change of x and u into $x - Vt$ and $u - V$, respectively. Thus establishing the adequate notion of solution and proving well-posedness and continuous dependence estimates for (1) is a building block for constructing solutions to the fluid-particle interaction model (2)–(3). Furthermore, our goal is to provide a simple numerical scheme for (1) (this means in particular that we want to avoid using the exact Riemann solver at the interface); such a solver can be the basis for an efficient numerical scheme for the coupled problem (2)–(3). We refer to the work [ALST10] of the authors with F. Lagoutière and T. Takahashi for further numerical motivations, and to [ALST10, ALST] for the application of the results of the present paper to the theoretical study of the model (2)–(3).

1.3 Notion of solution and well-posedness

It should be stressed that definition of solution to (1) is not straightforward; as a matter of fact, it requires a careful analysis carried out in the work [LST08] of Lagoutière, Takahashi and the second author. Indeed, the right-hand side of (1) involves a product of distributions, which is *a priori* not defined. In [LST08], the authors regularize the Dirac measure present in (1) and determine admissible left- and right-sided traces $\gamma_- u$, $\gamma_+ u$ of a solution u at $\{x = 0\}$. The regularization consists in approximation of the Dirac measure by the derivative of a locally smoothed Heavyside function, as initially introduced by LeRoux for shallow water equations with jumps of topography [LeR99] (see also [CLS04], [SV03] and [Bac05]). It is shown that if the smoothed function remains monotone, then the set $\mathscr{G}_\lambda \subset \mathbb{R}^2$ of possible one-sided traces $(\gamma_- u, \gamma_+ u)$ of u at $\{x = 0\}$ is independent of the choice of the regularization profile. The following definition of entropy solution is deduced:

$$\begin{array}{l} \text{an admissible solution to } \partial_t u + \partial_x(u^2/2) = -\lambda\, u\, \delta_0(x) \\ \text{is a Kruzhkov entropy solution away from the interface } \{x = 0\} \\ \text{such that the couple of left- and right-sided traces } (\gamma_- u, \gamma_+ u) \text{ of } u \\ \text{at } \{x = 0\} \text{ belongs to } \mathscr{G}_\lambda. \end{array} \tag{4}$$

Because existence of the one-sided traces of u on the interface is ensured by the result of Panov [Pan07], this notion of solution makes sense. In particular, one can deduce the existence and uniqueness of a solution of (1) for all Riemann data $u_0(x) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{\{x>0\}}$: the associated Riemann solver at the interface $\{x = 0\}$ is described in [LST08] (also a Riemann solver for the full problem (2)–(3) is described).

Using the theory of admissible germs developed by Karlsen, Risebro and the first author in [AKR11] (see also [AKR10]) in the context of conservation laws with discontinuous flux, we will prove uniqueness, comparison and L^1 contraction principle for (1), reformulate the above definition in terms of “adapted entropy” inequalities, and use this second formulation in order to prove existence.

1.4 A finite volume approach

Existence for (1) is justified by the classical approximation/compactness/passage-to-the-limit approach. Although different approximation strategies for proving existence can be employed (in particular, we mention in Section 4 that the LeRoux regularization method for (1) converges), we focus on a classical finite volume method with monotone and consistent numerical flux (see, e.g., [EGH00]), where we modify the flux only at the interface in such a way that the scheme remains well-balanced (namely, it preserves exactly some set of piecewise constant stationary solutions related to $\mathscr{G}_\lambda$). The modification is strikingly simple in view of the complexity of the interface Riemann solver for (1); and the efficiency of the scheme is illustrated by numerical experiments. The proof of convergence is rather short, it is based on the intrinsic properties of the germ $\mathscr{G}_\lambda$, on the well-balance property of the scheme, and on the local BV estimates obtained by the technique of Bürger et al. [BGKT08, BKT09]. Moreover, we hope that the insight from this scheme may carry on to the context of the full problem (2)–(3) and to the multi-dimensional case.

1.5 On balance laws with distributed source

Equation (1) can also be interesting in view of the approximation by the well-balanced Godunov scheme of the balance law

$$\partial_t u + \partial_x(u^2/2) = -u\; a'(x) \tag{5}$$

where a is a given non-decreasing function. Such a scheme is based on a cellwise constant discretization of a; and at each of so created interfaces, the Riemann problem associated with the system

$$\begin{cases} \partial_t u + \partial_x(u^2/2) + \lambda\, u \partial_x H = 0 \\ \partial_t H = 0 \end{cases} \tag{6}$$

must be solved, where $\lambda = a_{i+1} - a_i$ and $u_0(x) = u_i + (u_{i+1} - u_i)1\!\!1(x)$. Such equations have been deeply studied, and the associated Riemann problem is the cornerstone of the construction of well-balanced schemes for balance laws with a *smooth* source term: see in particular [GL96b, IT95, Gue04, AGG04] and also [BJ97, AP05, BPV03, GL04, Vas02, BCG08, Bou06] for related issues. In all these cases, the uniqueness of the solution of the Riemann problem fails and the only way to recover uniqueness for the Cauchy problem is to let $a_{i+1} - a_i$ tend to zero when $\Delta x \to 0$. As a consequence, the well-posedness and the convergence of the numerical schemes have been obtained in the only case of a smooth function a. In our case, the particular structure of the models (see Remark 2 for more details) enables to recover uniqueness and convergence of well-balanced schemes.

1.6 The outline of the paper

In Section 2, we give a description of the germs, which are the objects that govern the interface coupling for the problem in hand. Section 3 gives a description of the well-balanced scheme followed by consistency, stability and convergence analysis. Some complements are presented in Section 4 and in Appendices. Section 5 presents numerical illustrations for the scheme under study.

2 Germs and entropy solutions

The subject of this section is to provide a natural definition of solution for (1). This definition must be obtained using some modelling assumption, because the singular source term involves a product of distributions. Here, we propose to define this product as the limit of a regularization process, following [LST08].

2.1 Interface coupling and its properties

In order to define the behavior of the solution through the interface $\{x = 0\}$, we follow [LST08] where a regularization process is used (see also [LeR99], [SV03], [Vas02]). Actually, this method for defining solutions of PDE's with singular terms is equivalent to the one used in [IT95, GL04].

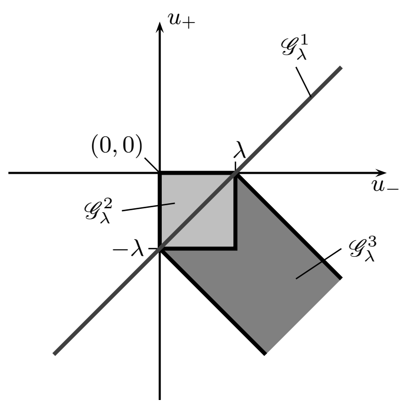

Figure 1: Representation of the admissiblity germ $\mathscr{G}_\lambda = \mathscr{G}^1_\lambda \cup \mathscr{G}^2_\lambda \cup \mathscr{G}^3_\lambda$.

The admissibility of weak solutions to (1) is governed by the *germ* $\mathscr{G}_\lambda$ (our terminology is related to the one of [AKR11]) defined as follows.

Definition 2.1. The *admissibility germ* $\mathscr{G}_\lambda \subset \mathbb{R}^2$ (or *germ*, for short) associated with (1) is defined as the union $\mathscr{G}_\lambda = \mathscr{G}^1_\lambda \cup \mathscr{G}^2_\lambda \cup \mathscr{G}^3_\lambda$, where

- $\mathscr{G}^1_\lambda = \{(a, a - \lambda), a \in \mathbb{R}\}$.
- $\mathscr{G}^2_\lambda = [0, \lambda] \times [-\lambda, 0]$.
- $\mathscr{G}^3_\lambda = \{(a, b) \in (\mathbb{R}^+ \times \mathbb{R}^-) \setminus \mathscr{G}^2_\lambda,\ -\lambda \leqslant a + b \leqslant \lambda\}$.

Notice that the partition of $\mathscr{G}_\lambda$ into the three parts is dictated by the subsequent analysis; while $\mathscr{G}^1_\lambda$ appears naturally in the proof of Proposition 2.2 below, the separation of $\mathscr{G}^2_\lambda$ and $\mathscr{G}^3_\lambda$ will become clear in Section 3.2.2.

In our interpretation that goes back to [LST08], the germ $\mathscr{G}_\lambda$ determines the interface coupling for problem (1). Indeed, let H denote the Heavyside function. In the sense of distributions, (1) is equivalent to

$$\partial_t u + \partial_x(u^2/2) = -\lambda u \partial_x H.$$

We introduce $H_\varepsilon \in \mathscr{C}^1(\mathbb{R})$ which is a non-decreasing function such that $H_\varepsilon(x) = H(x)$ when $|x| \geqslant \varepsilon$. Since we are interested in understanding the behavior of the solution through the stationary interface $\{x = 0\}$, we can study stationary solutions. We then obtain the regularized equation for $U_\varepsilon(x) = u(t,x)$:

$$(U_\varepsilon^2/2)'(x) + \lambda U_\varepsilon(x)\partial_x H_\varepsilon(x) = 0, \tag{7}$$

which has to be understood in the weak sense.

The following proposition is shown in [LST08]:

Proposition 2.2. *There exists a solution to ODE (7) with $U_\varepsilon(-\varepsilon) = a$ and $U_\varepsilon(-\varepsilon) = b$ if and only if $(a,b) \in \mathscr{G}_\lambda$.*

It is worth noting that the statement is independent from the choice of H_ε.

The **modelling assumption** we make is the following: the traces $\gamma_- u$ and $\gamma_+ u$ at $\{x = 0\}$ of a solution u of Problem (1) are *compatible* if and only if there exists an entropy weak solution to ODE (7) such that $U_\varepsilon(-\varepsilon) = \gamma_- u$, $U_\varepsilon(\varepsilon) = \gamma_+ u$ (these two boundary conditions must be understood in the strong sense). Thus we interprete Proposition 2.2 as the coupling admissibility condition, namely, the germ $\mathscr{G}_\lambda$ is the set of couples $(\gamma_- u, \gamma_+ u)$ of possible traces at $\{x = 0\}$ (for a.e. $t > 0$) of the admissible solutions of (1).

Now, the dissipativity properties of the interface coupling in equation (1) are encoded in the germ $\mathscr{G}_\lambda$. Indeed, let us define $\Xi\colon \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ by

$$\Xi\big((u_-,u_+),(v_-,v_+)\big) = \Phi(u_-,v_-) - \Phi(u_+,v_+) \tag{8}$$

where Φ is the so-called Kruzhkov entropy flux $\Phi(u,v) = \mathrm{sgn}(u-v)(u^2-v^2)/2$. Also set $\Phi^\pm(u,v) = \mathrm{sgn}^\pm(u-v)(u^2-v^2)/2$ and define $\Xi^\pm$ as in (8), using $\Phi^\pm$ in the place of Φ.

Splitting the germ $\mathscr{G}_\lambda$ into three subsets (see Definition 2.1 and Fig.2.1), we have

Proposition 2.3. *The following properties hold:*

(i) $\forall (u_-,u_+),(v_-,v_+) \in \mathscr{G}_\lambda$,
$\Xi\big((u_-,u_+),(v_-,v_+)\big) \geqslant 0,$ *moreover,* $\Xi^\pm\big((u_-,u_+),(v_-,v_+)\big) \geqslant 0.$

(ii) If a pair $(u_-,u_+) \in \mathbb{R}^2$ verifies:

$$\forall (v_-,v_+) \in \mathscr{G}_\lambda^1 \cup \mathscr{G}_\lambda^2 \quad \Xi\big((u_-,u_+),(v_-,v_+)\big) \geqslant 0, \tag{9}$$

then $(u_-,u_+) \in \mathscr{G}_\lambda$.

One can prove Proposition 2.3 directly, by a tedious case study; let us give a shorter but indirect proof in the spirit of [AKR11]. Indeed, in the terminology of [AKR11, AKR10], property (i) means that $\mathscr{G}_\lambda$ is a L^1-dissipative germ; and property (ii) means that $\mathscr{G}_\lambda^1 \cup \mathscr{G}_\lambda^2$ is a definite germ of which $\mathscr{G}_\lambda$ is the unique maximal extension.[1]

[1]To be precise, in [AKR11] the coupling was conservative, thus the Rankine-Hugoniot condition at the interface was included into the definition of a germ. Here we work with non-conservative coupling. Yet the definitions and the properties of germs stated in [AKR11] (see also [AKR10]) remain meaningful for a non-conservative coupling.

Proof. Property (i) (let us restrict our attention to Ξ^+) stems from the Kato inequality

$$\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R}), \varphi \geqslant 0,$$
$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big(\lambda (u^\varepsilon - v^\varepsilon)^+ \partial_x H_\varepsilon \, \varphi - (u^\varepsilon - v^\varepsilon)^+ \partial_t \varphi - \Phi^+(u^\varepsilon, v^\varepsilon) \partial_x \varphi \Big) \leqslant 0 \quad (10)$$

for entropy solutions $u^\varepsilon, v^\varepsilon$ of the regularized conservation law

$$\partial_t u + \partial_x (u^2/2) = -\lambda u \partial_x H_\varepsilon, \qquad (11)$$

$\varepsilon > 0$ (the Kato inequality is obtained from the standard Kruzhkov entropy formulation of (11) and the fact that $\lambda \partial_x H_\varepsilon \geqslant 0$, so that $\lambda u \partial_x H_\varepsilon$ acts as an absorption term). Indeed, it follows from Proposition 2.2 that the functions

$$u(t,x) := u_- 1\!\!1_{\{x<0\}} + u_+ 1\!\!1_{\{x>0\}}, \quad v(t,x) := v_- 1\!\!1_{\{x<0\}} + v_+ 1\!\!1_{\{x>0\}}$$

with $(u_-, u_+), (v_-, v_+) \in \mathscr{G}_\lambda$ can be obtained as $\mathbf{L}^1_{loc}$ limits, as $\varepsilon \to 0^+$, of stationary solutions of (11); thus we inherit the Kato inequality

$$-\int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big((u - v)^+ \, \partial_t \varphi + \Phi^+(u,v) \, \partial_x \varphi \Big) \leqslant 0 \;\; \forall \, \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R}), \, \varphi \geqslant 0.$$

Taking by approximation $\varphi(t,x) = \varphi_h(x)\psi(t)$ with $\psi \geqslant 0$ and with

$$\varphi_h(x) := 1 - \min\{1, \tfrac{|x|}{h}\}, \qquad (12)$$

letting $h \to 0^+$, we prove (i) for the quantity Ξ^+.

We have shown the germ $\mathscr{G}_\lambda$ is $\mathbf{L}^1$-dissipative. To prove property (ii) of Proposition 2.3, we point out that, according to the analysis of [LST08] (the case study is actually hidden in this result), all Riemann problem for (1) has a solution u with traces $(\gamma_- u, \gamma_+ u) \in \mathscr{G}_\lambda$; thus, in the terminology of [AKR11], the germ $\mathscr{G}_\lambda$ is complete. A complete $\mathbf{L}^1$-dissipative germ is maximal (see [AKR11, AKR10]); the maximality of $\mathscr{G}_\lambda$ exactly means that

$$\Big[\forall (v_-, v_+) \in \mathscr{G}_\lambda \quad \Xi\big((u_-, u_+), (v_-, v_+)\big) \geqslant 0 \Big] \implies \Big[(u_-, u_+) \in \mathscr{G}_\lambda \Big]. \qquad (13)$$

Property (ii) is stronger than (13), but it suffices to show the following:

$$\Big[\forall (v_-, v_+) \in \mathscr{G}^1_\lambda \cup \mathscr{G}^2_\lambda \quad \Xi\big((u_-, u_+), (v_-, v_+)\big) \geqslant 0 \Big]$$
$$\implies \Big[\forall (v_-, v_+) \in \mathscr{G}_\lambda \quad \Xi\big((u_-, u_+), (v_-, v_+)\big) \geqslant 0 \Big]. \qquad (14)$$

Indeed, from (13) and (14) one readily gets property (ii) .

In order to justify (14), given (u_-, u_+) that satisfies (9), we denote by $\widetilde{\mathscr{G}}$ the set of all $(v_-, v_+) \in \mathbb{R}^2$ such that $\Xi\big((u_-, u_+), (v_-, v_+)\big) \geqslant 0$. Because $\mathscr{G}^1_\lambda \cup \mathscr{G}^2_\lambda \subset \widetilde{\mathscr{G}}$, we only need to show that $\mathscr{G}^3_\lambda \subset \widetilde{\mathscr{G}}$. Take $(v_-, v_+) \in \mathscr{G}^3_\lambda$; in particular, $v_+ \leqslant 0 \leqslant v_-$. First, assume we also have $u_+ \leqslant 0 \leqslant u_-$; then $\Phi(u_-, v_-) \geqslant 0 \geqslant \Phi(u_+, v_+)$ because $z \mapsto \frac{z^2}{2}$ is increasing on $\mathbb{R}^+$ and decreasing on $\mathbb{R}^-$. Therefore in the case $u_+ \leqslant 0 \leqslant u_-$, we have $(v_-, v_+) \in \widetilde{\mathscr{G}}$.

Now we show that the boundary of $\mathscr{G}_\lambda^3$ is included in $\widetilde{\mathscr{G}}$; it suffices to get

$$\{(c_-,c_+)\,|\,c_+=-c_-+1,\ c_-\geqslant 1\}\cup\{(c_-,c_+)\,|\,c_+=-c_--1,\ c_-\geqslant 0\}\ \subset\ \widetilde{\mathscr{G}}.$$

Notice that, whenever (c_-,c_+) is in the above union of two sets, either the point $(c_-,-c_+)$ (when $c_+=-c_-+1$) or the point $(-c_-,c_+)$ (when $c_+=-c_--1$) belongs to $\mathscr{G}_\lambda^1$; this is easily seen on Fig. 2.1. Let us focus on the first case. Because $-c_+\geqslant 0\geqslant c_+$, we readily see that $-c_+1\!\!1_{\{x<0\}}+c_+1\!\!1_{\{x>0\}}$ and the constant u_+ are two Kruzhkov entropy solutions to the Burgers equation; it follows that $\Phi(u_+,-c_+)\geqslant\Phi(u_+,c_+)$. From the fact that $(c_-,-c_+)\in\mathscr{G}_\lambda^1\subset\widetilde{\mathscr{G}}$, we get $\Phi(u_+,-c_+)\geqslant\Phi(u_+,c_+)$; hence $\Phi(u_-,c_-)\geqslant\Phi(u_+,-c_+)\geqslant\Phi(u_+,c_+)$. Therefore $(c_-,c_+)\in\widetilde{\mathscr{G}}$. The second case is analogous.

Now we cut $\mathscr{G}_\lambda^3$ by the horizontal and by the vertical lines passing through (v_-,v_+). Considering the intersection points with $\partial(\mathscr{G}_\lambda^3)$, we see that there exist $v_+^{\flat,\sharp}$ (depending on v_-), $v_+^\flat\leqslant v_+\leqslant v_+^\sharp\leqslant 0$, such that $(v_-,v_+^\flat),(v_-,v_+^\sharp)$ belong to $\widetilde{\mathscr{G}}$; similarly, we can define $v_-^{\flat,\sharp}$ (depending on v_+) with $0\leqslant v_-^\flat\leqslant v_-\leqslant v_-^\sharp$, and $(v_+^{\flat,\sharp},v_+)\in\widetilde{\mathscr{G}}$. Let us show that if $u_+\geqslant 0$, $(v_-,v_+)\in\widetilde{\mathscr{G}}$. Indeed, in this case $\mathrm{sgn}(u_+-v_+^\flat)=\mathrm{sgn}(u_+-v_+^\sharp)$. Therefore, $\Phi(u_+,v_+)$ lies in between $\Phi(u_+,v_+^\flat)$ and $\Phi(u_+,v_+^\sharp)$. Thus the desired inequality $\Phi(u_-,v_-)\geqslant\Phi(u_+,v_+)$ follows as a convex combination of the true inequalities $\Phi(u_-,v_-)\geqslant\Phi(u_+,v_+^{\flat,\sharp})$. In conclusion, $(v_-,v_+)\in\widetilde{\mathscr{G}}$ unless $u_+<0$. Similarly, from consideration of $c_-^{\flat,\sharp}$ we see that $(v_-,v_+)\in\widetilde{\mathscr{G}}$ unless $u_->0$. But the case of $u_+<0<u_-$ has already been resolved; we conclude that in all the possible cases, $(v_-,v_+)\in\widetilde{\mathscr{G}}$.

This ends the proof of (14) and of property (ii). ☐

Remark 1. Unfortunately, statement (ii) of Proposition 2.3 is no longer true if we further weaken (13) by taking $(v_-,v_+)\in\mathscr{G}_\lambda^1$ instead of $(v_-,v_+)\in\mathscr{G}_\lambda^1\cup\mathscr{G}_\lambda^2$ in (9). This fact makes intricate the study of convergence of the numerical schemes we introduce in the sequel, because our schemes are well-balanced only with respect to the states in $\mathscr{G}_\lambda^1$. In the case of a quadratic source term presented in Appendix B, the set corresponding to $\mathscr{G}_\lambda^2$ can be omitted, which simplifies the analysis of the associated numerical scheme (see Appendix B for more details).

Remark 2. Uniqueness does not hold true when focusing on the linear drag force but with $\lambda<0$. Consider the equation $\partial_t u+\partial_x(u^2/2)-u\,\delta_0=0$ and the initial condition $u(0,x)=0$ for all $x\in\mathbb{R}$. Of course $u(t,x)=0$ for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$ is solution, but also

$$u(t,x)=\begin{cases}0 & \text{if } |x/t|>\alpha/2,\\ -\alpha & \text{if } -\alpha/2<x/t<0,\\ \alpha & \text{if } 0<x/t<\alpha/2,\end{cases}$$

for any $0<\alpha\leqslant 1/2$. On the other hand, let us recall that in the resonant case studied by Isaacson and Temple [IT95], up to three (self-similar) solutions may coexist.

2.2 Entropy solutions: definitions, properties, uniqueness

As pointed out in the Introduction, (4) is enough to define admissible entropy weak solutions of problem (1). This definition (together with Proposition 2.3(i))

readily yields uniqueness, comparison and continuous dependence properties for (1). Yet the explicit use of strong interface traces (see Panov [Pan07]) in (4) is a drawback, because the stability of such notion of admissible solution under the $\mathbf{L}^1_{loc}$ convergence is unclear. Thus we are heading towards another definition, or rather towards a series of equivalent definitions of admissible solutions to (1).

First, let us describe some elementary solutions of this problem: in the context of (1), they play the role of the constants in the standard Kruzhkov entropy formulation. Define piecewise constant functions c on $\mathbb{R}^+ \times \mathbb{R}$ by

$$c(t,x) = c_- 1\!\!1_{\{x<0\}} + c_+ 1\!\!1_{\{x>0\}} = \begin{cases} c_- & \text{if } x < 0, \\ c_+ & \text{if } x > 0, \end{cases} \tag{15}$$

where $c_-, c_+ \in \mathbb{R}$. Notice that c solves the Burgers equation away from the interface $\{x = 0\}$. Therefore according to the **modelling assumption** of Section 2.1, we consider that $c(t,x)$ is an admissible solution of (1) (with $u_0 \equiv c$) if and only if $(c_-, c_+) \in \mathscr{G}_\lambda$. As a consequence, making use of *adapted Kruzhkov entropies* (cf. [BJ97, AP05, AMVG05, BKT09, AKR11, AGS10]), we can provide a natural definition of entropy solution for Problem (1):

Definition 2.4. Consider the Cauchy problem (1) with $u_0 \in \mathbf{L}^\infty(\mathbb{R})$ and $\lambda > 0$. A function $u \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R})$ is said to be an *entropy solution* of (1) if for all function c defined by (15) with $(c_-, c_+) \in \mathscr{G}_\lambda$,

$$\forall \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^+ \times \mathbb{R}),\ \varphi \geqslant 0 \quad \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left[|u - c|\ \partial_t \varphi + \Phi(u,c)\ \partial_x \varphi \right]\, dx\, dt \\ + \int_{\mathbb{R}} |u_0 - c|\ \varphi(0,x)\ dx \geqslant 0. \tag{16}$$

Remark 3. Notice that if u takes the form $u(t,x) = u_- 1\!\!1_{\{x<0\}} + u_+ 1\!\!1_{\{x>0\}}$ with $(u_-, u_+) \in \mathscr{G}_\lambda$, then u is an entropy solution of (1) (this means that the elementary solutions that we have declared admissible are indeed entropy solutions). In fact, in this case, by a simple integration-by-parts (16) reduces to the inequality $\Xi\big((u_-, c_-), (u_+, c_+)\big) \int_{\mathbb{R}^+} \varphi(t,0)\, dt \geqslant 0$; this is true thanks to Proposition 2.3(ii) and because $\varphi \geqslant 0$.

Remark 4. In contrast with analogous definitions in [BKT09, AKR11, AKR10, AGS10], it may seem to the reader that the above definition is only concerned with the admissible coupling of $u|_{x<0}$ and $u|_{x>0}$ at the interface.

Yet it should be stressed that the above definition means in particular that u is a Kruzhkov entropy solution of the Burgers equation away from the interface $\{x = 0\}$. Indeed, it is sufficient to take φ with support in $\{x < 0\}$ or in $\{x > 0\}$, and notice that both c_- and c_+ describe the set of all real numbers when (c_-, c_+) describes the set $\mathscr{G}_\lambda$.

Remark 5. As soon as we know that u is a Kruzhkov entropy solution of the Burgers equation away from the interface $\{x = 0\}$, strong left- and right-side traces $\gamma_\pm u$ on $\{x = 0\}$ are well defined (see Vasseur [Vas01], Panov [Pan07]), because the Burgers flux $u \mapsto \frac{u^2}{2}$ is non-linear on every subinterval of $\mathbb{R}$.

Let us provide alternative characterizations of entropy solutions:

Proposition 2.5. *A function $u \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution if and only if it satisfies any of the following assertions:*

[A] *The function u verifies (16) for all non-negative function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ and for all function c defined by (15) with $(c_-, c_+) \in \mathscr{G}_\lambda^1 \cup \mathscr{G}_\lambda^2$.*

[B] *The function u verifies (4), i.e., the Kruzhkov entropy inequalities*

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \big[|u-\kappa|\ \partial_t\varphi + \Phi(u,\kappa)\ \partial_x\varphi\big]\ dx\ dt + \int_{\mathbb{R}} |u_0-\kappa|\ \varphi(0,x)\ dx \geqslant 0 \quad (17)$$

hold for all $\kappa \in \mathbb{R}$ and for all non-negative test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ such that $\varphi|_{x=0} = 0$, moreover,

$$\textit{for a. e. } t > 0 \ \ \big(\,(\gamma_- u)(t)\,,\,(\gamma_+ u)(t)\,\big) \in \mathscr{G}_\lambda. \quad (18)$$

[C] *The function u verifies (17) for all $\kappa \in \mathbb{R}$ and all non-negative function φ of $\mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R})$, moreover, for all $(c_-, c_+) \in \mathscr{G}_\lambda$ there holds*

$$\textit{for a.e. } t>0 \ \ \Xi\Big(\big((\gamma_- u)(t), (\gamma_+ u)(t)\big)\,,\,(c_-, c_+)\Big) \geqslant 0. \quad (19)$$

[D] *There exists $C = C(\lambda, \|u\|_\infty, c_\pm) > 0$ such that the function u verifies*

$$\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R}),\ \varphi \geqslant 0 \quad \int_{\mathbb{R}^+}\int_{\mathbb{R}} \big[|u-c|\ \partial_t\varphi + \Phi(u,c)\ \partial_x\varphi\big]\ dx\ dt$$
$$+ \int_{\mathbb{R}} |u_0 - c|\ \varphi(0,x)\ dx \geqslant -C\, \mathrm{dist}\Big((c_-, c_+)\,,\,\mathscr{G}_\lambda\Big) \quad (20)$$

for all function c defined by (15) with $(c_-, c_+) \in \mathbb{R} \times \mathbb{R}$.

Characterization [A] will be used to prove convergence of the numerical schemes presented in the next section, while characterization [B] will be used for the proof of uniqueness. Characterization [B] is most intuitive, it goes back to the modelling process of [LST08] and Section 2.1. Formulation [C] describes explicitly the interface dissipation property of entropy solutions (the reader may check that $\mathscr{G}_\lambda$ can be replaced with $\mathscr{G}_\lambda^1 \cup \mathscr{G}_\lambda^2$ in the statement of [C]). Finally, characterization [D] is a convenient generalization of Definition 2.4, it allows to treat equation (2) of the full particle-in-Burgers model (see [ALST10, ALST]).

Proof. We show that Definition 2.4 $\Rightarrow$ [A] $\Rightarrow$ [B] $\Rightarrow$ [C] $\Rightarrow$ [D] $\Rightarrow$ Definition 2.4.

Firstly, Definition 2.4 clearly implies [A].

Next, as in Remark 4, inequalities (16) of [A] still imply the Kruzhkov inequalities (17). Moreover, using the test functions φ_h of the form (12), for all $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^+)$, $\psi \geqslant 0$ we derive as $h \to 0^+$ the inequalities

$$\forall\, (c_-, c_+) \in \mathscr{G}_\lambda^1 \cup \mathscr{G}_\lambda^2 \quad \int_{\mathbb{R}^+} \Big(\Phi\big((\gamma_- u)(t), c_-\big) - \Phi\big((\gamma_+ u)(t), c_+\big)\Big)\, \psi(t)\, dt \geqslant 0.$$

Because ψ is arbitrary, making ψ concentrate at a Lebesgue point of the maps $t \mapsto (\gamma_\pm u)(t)$, we derive that (19) holds with $(c_-, c_+) \in \mathscr{G}_\lambda^1 \cup \mathscr{G}_\lambda^2$. By a simple application of Proposition 2.3(ii), we deduce (18). Thus [C] holds.

Applying Proposition 2.3(i) one gets (19) from (18); thus [B] implies [C].

To get from [C] to Definition 2.4 and to [D], we split the test function φ as $\varphi\varphi_h + \varphi(1 - \varphi_h)$, where φ_h is defined by (12). Then we can write inequalities

(17) with the test function $\varphi(1-\varphi_h)$ which is zero on $\{x=0\}$; the difference of the left-hand side of this inequality with the left-hand side of (20) is the term

$$R_h := \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left[|u-c|\;\partial_t(\varphi\varphi_h) + \Phi(u,c)\;\partial_x(\varphi\varphi_h)\right]\,dx\,dt + \int_{\mathbb{R}} |u_0-c|\;\varphi(0,x)\varphi_h(x).$$

From the definition of φ_h and of the strong interface traces $\gamma_\pm u$, we compute

$$\lim_{h\to 0^+} R_h = \int_{\mathbb{R}^+} \Xi\Big(\big((\gamma_- u)(t),(\gamma_+ u)(t)\big)\,,\,(c_-,c_+)\Big)\,\varphi(t,0)\,dt.$$

Denote $\big((\gamma_- u)(t),(\gamma_+ u)(t)\big) = (a_-,a_+)$. By (19), if we have $(c_-,c_+)\in\mathscr{G}_\lambda$, we find $\Xi\big((a_-,a_+)\,,\,(c_-,c_+)\big)\geqslant 0$; hence inequalities (16) of Definition 2.4 follow. More generally, for $(c_-,c_+)\in\mathbb{R}\times\mathbb{R}$, we first take a closest to (c_-,c_+) element of $\mathscr{G}_\lambda$; denote it by (b_-,b_+). Then

$$\begin{aligned}\Xi\big((a_-,a_+),(c_-,c_+)\big) \geqslant\; &\Xi\big((a_-,a_+),(b_-,b_+)\big)\\ &- \Big|\Xi\big((a_-,a_+),(c_-,c_+)\big) - \Xi\big((a_-,a_+),(b_-,b_+)\big)\Big|,\end{aligned}$$

where the first term is non-negative by (19), and the second one is dominated by a constant $C = C(|a_\pm|,|b_\pm|,|c_\pm|)$ times the distance from (c_-,c_+) to (b_-,b_+). Hence we get [D], because $|a_\pm|\leqslant\|u\|_\infty$ and $|b_\pm|$ is estimated via $|c_\pm|$ and λ.

Finally, [D] trivially implies inequalities (16) of Definition 2.4. $\square$

We are now in a position to state the result of $\mathbf{L}^1$ contraction, comparison and uniqueness for entropy solutions of (1).

Theorem 2.6. *Let u_0 and v_0 be two initial data in $\mathbf{L}^\infty(\mathbb{R})$ and let u and v be the associated entropy solutions of Problem (1). Then for all $R>0$,*

$$\text{for a.e. } t>0 \quad \int_R^R (u-v)^+(t,x)\;dx \leqslant \int_{-R-Lt}^{R+Lt} (u_0-v_0)^+(x)\;dx \tag{21}$$

where $L=\max\{\|u\|_\infty,\|v\|_\infty\}$. Consequently, if $(u_0-v_0)^+\in\mathbf{L}^1(\mathbb{R})$, we have

$$\text{for a.e. } t>0 \quad \int_{\mathbb{R}} (u-v)^+(t,x)\;dx \leqslant \int_{\mathbb{R}} (u_0-v_0)^+(x)\;dx. \tag{22}$$

In particular, for all $u_0\in\mathbf{L}^\infty(\mathbb{R})$, there exists at most one solution of Problem (1); and the map $\mathcal{S}(t): u_0\mapsto u(t,\cdot)$ on its domain is an order-preserving contraction for the $\mathbf{L}^1(\mathbb{R})$ norm.

Later we will also prove existence of solutions to (1) for all $\mathbf{L}^\infty$ datum u_0.

Proof. The arguments of the proof are standard in the framework of conservation laws with discontinuous fluxes (see for instance [AMVG05, BKT09, AKR11, AGS10]). Let us only provide a sketch of the proof.

Using the Kruzhkov inequalities with $\kappa=\pm\|u\|_\infty$ we can replace $|u-\kappa|$ and $\Phi(u,\kappa)$ by $(u-\kappa)^\pm$ and $\Phi^\pm(u,\kappa)$, respectively. Then with the classical doubling of variables technique of Kruzhkov [Kru70] with non-negative test functions

$\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ such that $\varphi = 0$ on $\{x = 0\}$, we get the Kato inequality for two entropy solutions u and v of (1):

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big((u-v)^+ \partial_t \varphi + \Phi^+(u,v)\partial_x \varphi \Big) \; dx\, dt + \int_{\mathbb{R}} (u_0 - v_0)^+ \varphi(0,x)\; dx \geqslant 0. \quad (23)$$

Take $R > 0$; set $L = \max\{\|u\|_\infty, \|v\|_\infty\}$ (then L bounds the speed of characteristics of the Burgers equation associated with solutions u and v). Introducing the test function $\varphi\varphi_h$ with φ_h of the form (12) and with φ approximating the characteristic function of $\{(s,x)\,|\,0 \leqslant s \leqslant t,\ |x| \leqslant R + L(t-s)\}$, from (23) and the strong trace properties we get for a.e. $t > 0$

$$- \int_{-R}^{R} (u-v)^+(t,x)\; dx + \int_{-R-LT}^{R+LT} (u_0 - v_0)^+(x)\; dx \geqslant \int_0^T \Xi^+\big((\gamma_- u(t), \gamma_+ u(t)), (\gamma_- v(t), \gamma_+ v(t))\big)\; dt. \quad (24)$$

The pairs $(\gamma_- u(t), \gamma_+ u(t))$ and $(\gamma_- v(t), \gamma_+ v(t))$ both belong to $\mathscr{G}_\lambda$, according to the characterization [B] of entropy solutions to Problem (1). Using Proposition 2.3(ii), we deduce that the right-hand side in (24) is non-negative. This yields the conclusion (21); then (22) follows as $R \to \infty$. ☐

As a first application, let us give an $\mathbf{L}^\infty$ bound on a entropy solution of (1) in terms of $\|u_0\|_\infty$ and λ (notice that the maximum principle fails for problem (1); indeed, according to [LST08] any constant data $u_0 \equiv c \neq 0$ lead to a non-constant solution of the Riemann problem). In particular, L in Theorem 2.6 can be bounded by the explicit constant $\lambda + \max\{\|u_0\|_\infty, \|v_0\|_\infty\}$.

Proposition 2.7. *Let $u_0 \in \mathbf{L}^\infty(\mathbb{R})$ and let u be the associated entropy solution of Problem (1). Then for a.e. $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$,*

$$\min\{\operatorname*{ess\,inf}_{\mathbb{R}^-} u_0 - \lambda\,,\ \operatorname*{ess\,inf}_{\mathbb{R}^+} u_0\} \leqslant u(t,x) \leqslant \max\{\operatorname*{ess\,sup}_{\mathbb{R}^-} u_0\,,\ \operatorname*{ess\,sup}_{\mathbb{R}^+} u_0 + \lambda\} \quad (25)$$

Proof. Let us prove, e.g., the lower bound for u. If c_+ denotes the left-hand side of (25) and $c_- := c_+ + \lambda$, then $(c_-, c_+) \in \mathscr{G}_\lambda^1$ and (16) holds true; moreover, a.e. on $\mathbb{R}$ we have $u_0(x) \geqslant c(x) := c_- 1\!\!1_{\{x<0\}} + c_+ 1\!\!1_{\{x>0\}}$.

By Remark 3, $c(\cdot)$ is a stationary entropy solution of (1). Therefore we can apply the comparison property (22) to find that $c(x) \leqslant u(t,x)$ a.e. on $\mathbb{R}^+ \times \mathbb{R}$; in addition, λ being positive, from the definition of $c(\cdot)$ and c_- we have $c_+ \leqslant c(x)$ for a.a. x. Thus $c_+ \leqslant u(t,x)$ a.e.. ☐

3 Well-balanced finite volume schemes

We now focus on the numerical approximation of the solution of (1). We will construct very simple numerical schemes which are able to maintain particular steady states – they are *well-balanced schemes*. These numerical schemes do not make use of any Riemann solver and converge to the entropy solution of (1) (this provides the existence of the entropy solution). The following analysis relies on *a priori* estimates in $\mathbf{L}^\infty$ (of the same kind as in Proposition 2.7) and in $\mathrm{BV}_{\mathrm{loc}}$ (following [BGKT08, BKT09]). Unfortunately, we need to impose an additional (and probably, purely technical) assumption on the numerical flux. In practice, this assumption is not restrictive; it is fulfilled by standard monotone schemes: Rusanov, Godunov, Engquist-Osher (see Appendix A).

Let us now define the generic form of the numerical schemes we propose. For the sake of notational simplicity, we only consider uniform meshes; the non-uniform case is analogous, provided the CFL condition is ensured. Thus, let Δx be the (positive) space step and define the interfaces of the mesh by $x_{i+1/2} = i\Delta x$ (the source term with the Dirac measure is then localized at the interface $x_{1/2}$). We also introduce the time step Δt which will be subject to a CFL condition related to the L^∞ estimate of Proposition 2.7, and set $t^n = n\Delta t$. We use a classical finite volume discretization approach: we discretize the initial data by

$$\forall i \in \mathbb{Z} \qquad u_i^0 = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u_0(x)\ dx \tag{26}$$

and look for discrete unknowns u_i^n intended to approximate u near (t^n, x_i), $x_i = (x_{i+1/2} + x_{i-1/2})/2$:

$$\forall i \in \mathbb{Z}, n \in \mathbb{N} \qquad u_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(t^n, x)\ dx \tag{27}$$

(a precise meaning can be given to (27) after the convergence result of Theorem 3.9 is established).

Starting from $(u_i^0)_i$ given by (26), the sequence of approximations $(u_i^n)_{i,n}$ is defined recursively by

$$\forall i \neq 0, 1 \qquad u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)); \tag{28}$$

$$u_0^{n+1} = u_0^n - \frac{\Delta t}{\Delta x}(g_\lambda^-(u_0^n, u_1^n) - g(u_{-1}^n, u_0^n)); \tag{29}$$

$$u_1^{n+1} = u_1^n - \frac{\Delta t}{\Delta x}(g(u_1^n, u_2^n) - g_\lambda^+(u_0^n, u_1^n)). \tag{30}$$

Here $g : \mathbb{R}^2 \to \mathbb{R}$ is a numerical flux (see e.g. [EGH00]); as usual, we require that g is locally Lipschitz, consistent (with the Burgers flux $u \mapsto u^2/2$), and monotone (nondecreasing w.r.t. its first variable and nonincreasing w.r.t. its second variable). Further, the two numerical fluxes $g_\lambda^\pm$ must account for the contribution of the singular source term in equation (1) at $x = 0$; they will be described in the next section. Let us just stress here that the consistency of such scheme with equation (1) is not a straightforward issue. One obvious possibility would be to use, at the interface, the Godunov scheme obtained from the Riemann solver described in [LST08]. We will see that a simpler choice exists.

To condense the calculations, in the sequel we adopt the notation

$$\forall i \in \mathbb{Z} \quad u_i^{n+1} = H_i(u_{i-1}^n, u_i^n, u_{i+1}^n), \tag{31}$$

where H_i is defined by (28) for $i \neq 1, 0$, by (29) for $i = 0$, and by (30) for $i = 1$.

3.1 Reconstructed states and well-balanced properties

The basic idea of the well-balanced schemes is the exact preservation at the numerical level of some selected piecewise constant stationary solutions [GL96b, GL96a]. According to the analysis of Section 2.1, the stationary solutions that are of utmost importance here are of the form $u(t,x) = c(x)$ with $c(\cdot)$ given by (15), where the pair (c_-, c_+) belongs to $\mathscr{G}_\lambda$. We will see now that $\mathscr{G}_\lambda$ furnishes "too many" stationary solutions of this form. The well-balance property becomes simpler to achieve if only solutions corresponding to a part $\mathscr{G}_\lambda^0$ of the germ $\mathscr{G}_\lambda$ are preserved by the scheme.

Let us insert such a function $c(\cdot)$ with $(c_-, c_+) \in \mathscr{G}_\lambda$ as initial datum into (29) and (30); assuming the well-balance property, the values u_i^n do not change with n, and we obtain (using the consistency of g)

$$c_- = c_- - \frac{\Delta t}{\Delta x}(g_\lambda^-(c_-, c_+) - (c_-)^2/2) \tag{32}$$

$$c_+ = c_+ - \frac{\Delta t}{\Delta x}((c_+)^2/2 - g_\lambda^+(c_-, c_+)). \tag{33}$$

Again, because the flux g is consistent the preservation of the values for $i \neq 0, 1$ is evident. This yields the following sufficient condition for preservation of $c(\cdot)$:

$$\forall (c_-, c_+) \in \mathscr{G}_\lambda^0 \quad \begin{cases} g_\lambda^-(c_-, c_+) = (c_-)^2/2 \\ g_\lambda^+(c_-, c_+) = (c_+)^2/2. \end{cases} \tag{34}$$

In order to make simpler the implementation of the scheme (28-30), we would like that the fluxes $g_\lambda^\pm$ be defined using the flux g. We look for the ansatz

$$g_\lambda^-(a,b) = g(a, \varphi_\lambda^-(b)) \quad \text{and} \quad g_\lambda^+(a,b) = g(\varphi_\lambda^+(a), b). \tag{35}$$

The numerical flux g being consistent with the Burgers flux, it is easy to comply with condition (34) if $\varphi_\lambda^\pm$ satisfies

$$\forall (c_-, c_+) \in \mathscr{G}_\lambda^0 \quad \varphi_\lambda^-(c_+) = c_- \quad \text{and} \quad \varphi_\lambda^+(c_-) = c_+. \tag{36}$$

Because $\mathscr{G}_\lambda$ is not the graph of a bijective map, we are not able to define $\varphi_\lambda^\pm$ if $\mathscr{G}_\lambda^0 := \mathscr{G}_\lambda$ (such a definition would lead to multivalued functions). Therefore, we focus on the choice $\mathscr{G}_\lambda^0 := \mathscr{G}_\lambda^1$, which is a bijective graph. Eventually, we define the *reconstructed states* as follows:

$$\varphi_\lambda^\pm(a) = a \mp \lambda. \tag{37}$$

We see that the definition of $g_\lambda^\pm$ via (35), (37) leads to the following property:

Proposition 3.1. *Consider the numerical scheme (28-30) with*

$$g_\lambda^-(a,b) = g(a, b+\lambda) \quad \textit{and} \quad g_\lambda^+(a,b) = g(a-\lambda, b). \tag{38}$$

Then any initial datum $u_0 \equiv c$ with c given by (15) and $(c_-, c_+) \in \mathscr{G}_\lambda^1$ discretized as in (26) is exactly preserved:

$$\forall n \in \mathbb{N} \quad u_i^n = \begin{cases} c_- & \textit{if } i \leqslant 0 \\ c_+ & \textit{if } i > 0 \end{cases}. \tag{39}$$

In other words, the numerical scheme (28-30)-(38) is *well-balanced* w.r.t. $\mathscr{G}_\lambda^1$. Note that Proposition 3.1 can also be seen as a basic consistency property, in the same way as consistent monotone schemes for conservation laws exactly preserve constant solutions.

3.2 Analysis of the well-balanced schemes

Let us now focus on the analysis of the numerical scheme (28-30)-(38). In all the following, the time step Δt agrees with the following CFL condition:

$$2M\Delta t \leqslant \Delta x \tag{40}$$

where M is the Lipschitz constant of the numerical flux g, e.g. on the interval $[\operatorname{ess\,inf}_{\mathbb{R}} u_0 - \lambda, \operatorname{ess\,sup}_{\mathbb{R}} u_0 + \lambda]$ (cf. Proposition 2.7).

It is well known that under the CFL condition (40) with M being a Lipschitz constant for a monotone numerical flux g, for $i \neq 0, 1$ the functions H_i in (31) are monotone non-decreasing in each of the three arguments. Since $\cdot \mapsto \cdot \pm \lambda$ are increasing monotone functions, also for $i = 0$ and $i = 1$ the monotonicity holds. Thus, as far as (40) holds and the values u_n^i stay in the prescribed interval, the numerical scheme (28-30)-(38) for $(u_i^{n+1})_i$ is monotone, *i.e.*

$$H_i \text{ in (31) is a nondecreasing function of each of its three arguments.} \tag{41}$$

3.2.1 Stability

Let us begin with the $\mathbf{L}^\infty$ bounds that justify the applicability of the CFL condition (40):

Lemma 3.2. *Under the CFL condition (40), the numerical scheme (28-30)-(38) satisfies for all $n \in \mathbb{N}$ and $i \in \mathbb{Z}$*

$$\min\{\operatorname*{ess\,inf}_{\mathbb{R}^-} u_0 - \lambda\,,\, \operatorname*{ess\,inf}_{\mathbb{R}^+} u_0\} \leqslant u_i^n \leqslant \max\{\operatorname*{ess\,sup}_{\mathbb{R}^-} u_0\,,\, \operatorname*{ess\,sup}_{\mathbb{R}^+} u_0 + \lambda\}.$$

Proof. The proof is the same as for Proposition 2.7; it stems from the comparison argument with the stationary solution of our well-balanced scheme given by (39), $(c_-, c_+) \in \mathscr{G}_\lambda^1$, with the same choice of $c_\pm$ as in Proposition 2.7. We proceed by induction in n; for $n = 0$ the claim holds. At each step, we use the monotonicity (41) of H to derive the inequality $u_i^{n+1} \geqslant c_i := c_- 1\!\!1_{i\leqslant 0} + c_+ 1\!\!1_{i \geqslant 1}$. □

Next, we need some compactness techniques allowing for the passage to the limit in the nonlinear scheme. It is easy to prove BV estimates on finite volume approximations of the Burgers equation; but the action of the singular source term at $\{x = 0\}$ makes delicate to find uniform BV bounds on the numerical scheme for (1). Indeed, it is easily seen that the scheme typically produces non-constant solutions from constant initial datum. We will investigate global BV bounds for solutions of (1) in the forthcoming work [ALST] with Lagoutière and Takahashi. As alternatives, one can use compactification arguments due to the nonlinearity of the Burgers flux, or seek to bypass strong compactness arguments by using measure-valued (entropy-process) solutions (see [EGH00]; cf. [AGS10], in the context of discontinuous flux). Our approach is to deduce compactness from $\mathrm{BV}_{\mathrm{loc}}$ estimates combined the Cantor's diagonal argument. Indeed, uniform $\mathrm{BV}_{\mathrm{loc}}$ bounds on any interval which does not intersect with $\{x = 0\}$ were obtained by Bürger *et al.* in [BGKT08, BKT09], in the context of conservation laws with discontinuous flux; we reproduce their approach. Unfortunately, here and also in Proposition 3.8 (discrete Kato inequalities) we need the following additional assumption on the numerical flux g:

$$g(\cdot + \lambda, b+\lambda) - g(\cdot, b) \;\text{ and }\; g(a+\lambda, \cdot + \lambda) - g(a, \cdot) \;\text{ are non-decreasing.} \tag{\textbf{H0}}$$

This condition ensures that the scheme is *dissipative at the interface*. It is easily seen that a sufficient condition for (**H0**) is

$$\partial_a(\partial_a g(a,b) + \partial_b g(a,b)) \geqslant 0 \quad \text{and} \quad \partial_b(\partial_a g(a,b) + \partial_b g(a,b)) \geqslant 0; \tag{\textbf{H}}$$

moreover, if we want (**H0**) to hold for every $\lambda > 0$, then (**H**) is an equivalent requirement. Thus, we stick to assumption (**H**) in the sequel. Note that the most classical numerical fluxes (Godunov, Rusanov, Engquist-Osher) fulfill Assumption (**H**) (see Appendix A).

Lemma 3.3 (BV_t bounds)**.** *Assume $u_0 \in \mathrm{BV}(\mathbb{R})$. For $T > 0$, consider $N \in \mathbb{N}$ and $\Delta t > 0$ such that $N\Delta t \leqslant T$. Under CFL condition (40) and assumption* (**H**)*, the numerical scheme (26)-(28-30)-(38) satisfies for all $n \in \mathbb{N}$*

$$\Delta x \sum_{n=0}^{N-1} \sum_{i\in\mathbb{Z}} |u_i^{n+1} - u_i^n| \leqslant K \tag{42}$$

where K is a positive constant only depending on $|u_0|_{\mathrm{BV}(\mathbb{R})}$, T , M and λ.

In the conservative case, the Crandall-Tartar lemma (see [CT80]) is sufficient to provide (42), without assumption (**H**). Here, a remaining term at the interface $x_{1/2}$ must have the good sign in order to obtain (42) (see Lemma 3.4).

Proof. We use the notation $\perp$ (resp. $\top$) to denote the min (resp., the max) of two elements. With this notation, we have

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} \big|u_i^{n+1} - u_i^n\big| \\
&\quad = \sum_{i\in\mathbb{Z}} \big(u_i^{n+1} - u_i^n\big)^+ + \sum_{i\in\mathbb{Z}} \big(u_i^n - u_i^{n+1}\big)^+ \\
&\quad = \sum_{i\in\mathbb{Z}} \big[u_i^{n+1} \top u_i^n - u_i^n\big] + \sum_{i\in\mathbb{Z}} \big[u_i^{n+1} \top u_i^n - u_i^{n+1}\big] \\
&\quad = \sum_{i\in\mathbb{Z}} \big[H_i(u_{i-1}^n, u_i^n, u_{i+1}^n) \top H_i(u_{i-1}^{n-1}, u_i^{n-1}, u_{i+1}^{n-1}) - H_i(u_{i-1}^{n-1}, u_i^{n-1}, u_{i+1}^{n-1})\big] \\
&\quad + \sum_{i\in\mathbb{Z}} \big[H_i(u_{i-1}^n, u_i^n, u_{i+1}^n) \top H_i(u_{i-1}^{n-1}, u_i^{n-1}, u_{i+1}^{n-1}) - H_i(u_{i-1}^n, u_i^n, u_{i+1}^n)\big] \\
&\quad \leqslant \sum_{i\in\mathbb{Z}} \big[H_i(u_{i-1}^n \top u_{i-1}^{n-1}, u_i^n \top u_i^{n-1}, u_{i+1}^n \top u_{i+1}^{n-1}) - H_i(u_{i-1}^{n-1}, u_i^{n-1}, u_{i+1}^{n-1})\big] \\
&\quad + \sum_{i\in\mathbb{Z}} \big[H_i(u_{i-1}^n \top u_{i-1}^{n-1}, u_i^n \top u_i^{n-1}, u_{i+1}^n \top u_{i+1}^{n-1}) - H_i(u_{i-1}^n, u_i^n, u_{i+1}^n)\big];
\end{aligned}$$

here monotonicity (41) of H_i is used, due to the CFL condition (40). At this stage, Crandall and Tartar [CT80] use the conservativity of the scheme, which fails here at the interface $x_{1/2}$. Yet it is easy to generalize the Crandall-Tartar lemma under the sub-conservativity condition; namely, in the notation of the Crandall and Tartar [CT80], one replaces the conservativity by the assumption

$$f \geqslant g \implies \int (T(f) - T(g)) \leqslant \int (f - g). \tag{43}$$

For the sake of completeness, we will prove this version, for the concrete map T given by $(H_i)_{i\in\mathbb{Z}}$. Condition (43) is available thanks to the following lemma.

Lemma 3.4. *Consider two sequences $(u_i)_{i\in\mathbb{Z}}$ and $(v_i)_{i\in\mathbb{Z}}$. Assume that $u_i \geqslant v_i$ for all $i \in \mathbb{Z}$. Let H_i be the fluxes defined in the beginning of the section (in particular, (38) is used). Then, under assumption (**H**),*

$$\sum_{i\in\mathbb{Z}} \big(H_i(u_{i-1}, u_i, u_{i+1}) - H_i(v_{i-1}, v_i, v_{i+1})\big) \leqslant \sum_{i\in\mathbb{Z}} (u_i - v_i) \tag{44}$$

Proof of Lemma 3.4. We have by definition of H_i (including the conservativity of the fluxes except for the interface $x_{1/2} = 0$ and the choice (38) of $g_\lambda^\pm$)

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} \big(H_i(u_{i-1}, u_i, u_{i+1}) - H_i(v_{i-1}, v_i, v_{i+1})\big) - \sum_{i\in\mathbb{Z}} (u_i - v_i) \\
&\quad = \frac{\Delta t}{\Delta x}(-g_\lambda^-(u_0, u_1) + g_\lambda^+(u_0, u_1) + g_\lambda^-(v_0, v_1) - g_\lambda^+(v_0, v_1)) \\
&\quad = \frac{\Delta t}{\Delta x}(-g(u_0, u_1 + \lambda) + g(u_0 - \lambda, u_1) + g(v_0, v_1 + \lambda) - g(v_0 - \lambda, v_1)).
\end{aligned}$$

Since $\lambda > 0$ and $u_i \geqslant v_i$ $\forall i \in \mathbb{Z}$, the right-hand side is non-positive by assumption (**H**), which gives (44). □

Continuing the proof of Lemma 3.3, we put $u_i = u_i^n \top u_i^{n-1}$ with $v_i = u_i^n$ and then with $v_i = u_i^{n-1}$ in (44). We obtain by induction in n,

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} \big|u_i^{n+1} - u_i^n\big| &\leqslant \sum_{i\in\mathbb{Z}} \big[u_i^n \top u_i^{n-1} - u_i^{n-1}\big] + \sum_{i\in\mathbb{Z}} \big[u_i^n \top u_i^{n-1} - u_i^n\big] \\
&\leqslant \sum_{i\in\mathbb{Z}} \big(u_i^n - u_i^{n-1}\big)^+ + \sum_{i\in\mathbb{Z}} \big(u_i^{n-1} - u_i^n\big)^+ \\
&= \sum_{i\in\mathbb{Z}} \big|u_i^n - u_i^{n-1}\big| \leqslant \quad \ldots \quad \leqslant \sum_{i\in\mathbb{Z}} \big|u_i^1 - u_i^0\big|.
\end{aligned}$$

The right-hand side can be easily estimated: by (28-30), we have

$$\begin{aligned}
\frac{\Delta x}{\Delta t} \sum_{i\in\mathbb{Z}} \big|u_i^1 - u_i^0\big| \leqslant &\sum_{i\neq 0,1} |g(u_i^0, u_{i+1}^0) - g(u_{i-1}^0, u_i^0)| \\
&+ |g(u_0^0, u_1^0 + \lambda) - g(u_{-1}^0, u_0^0)| + |g(u_1^0, u_2^0) - g(u_0^0 - \lambda, u_1^0)| \\
&\leqslant 2M(|u_0|_{\mathrm{BV}(\mathbb{R})} + \lambda).
\end{aligned}$$

Thus for $T \geqslant N\Delta t$, we get the required estimate

$$\begin{aligned}
\Delta x \sum_{n=0}^{N-1} \sum_{i\in\mathbb{Z}} |u_i^{n+1} - u_i^n| &\leqslant 2N\Delta t\, M(|u_0|_{\mathrm{BV}(\mathbb{R})} + \lambda) \\
&\leqslant 2TM(|u_0|_{\mathrm{BV}(\mathbb{R})} + \lambda) \quad =: K,
\end{aligned}$$

which concludes the proof. □

Let us introduce the classical notation

$$u_\Delta(t, x) = \sum_{n\in\mathbb{N}} \sum_{i\in\mathbb{Z}} u_i^n \; 1\!\!1_{(n\Delta t, (n+1)\Delta t)}(t) \; 1\!\!1_{(x_{i-1/2}, x_{i+1/2})}(x). \tag{45}$$

We are in position to provide $\mathrm{BV}_{\mathrm{loc}}$ (in space) bounds, following Lemma 4.2 of [BGKT08] (see also [BKT09]):

Lemma 3.5. *Let $T > 0$ and $A > 0$. Assume that $u_0 \in \mathrm{BV}(\mathbb{R})$ and Δx is small enough. Then, under the CFL condition (40) and assumption (**H**), the numerical scheme satisfies*

$$|u_\Delta(T,.)|_{\mathrm{BV}([A,+\infty))} \leqslant |u_0|_{\mathrm{BV}([A,+\infty))} + \frac{K}{A} \tag{46}$$

with K only depending on $|u_0|_{\mathrm{BV}(\mathbb{R})}$, T , M and λ.

Analogous results hold on the interval $(-\infty, -A]$.

Proof. The idea is to use the uniform $BV((0,T); L^1(\mathbb{R}))$ estimate of the functions u_Δ (given by Lemma 3.3) to deduce $BV(0,T)$ estimates on the functions $u_\Delta(\cdot, y_\Delta)$ with $y_\Delta \in (0,A)$ given by the mean value theorem. Then we interprete the solution of our finite volume scheme as being the solution to the monotone scheme of the Cauchy-Dirichlet problem set up in $[y_\Delta, +\infty) \times [0,T]$; both initial and Dirichlet data have controlled variation. By the well known results for the Cauchy-Dirichlet problem, this yields bound (46). We refer to [BGKT08, BKT09] for details. ☐

3.2.2 Consistency

We now investigate consistency of the well-balanced schemes with the entropy inequalities (16). As far as monotone schemes and conservation laws are concerned, discrete entropy inequalities are obtained and thanks to a Lax-Wendroff kind theorem, one obtains the "continuous" entropy inequalities passing to the limit. One of the main ingredients to obtain discrete entropy inequalities is the monotonicity (41) of the scheme. Another ingredient, in the classical setting, is the preservation of constant solutions. In our framework, the same role is played by the solutions (15) with $(c_-, c_+) \in \mathscr{G}$ (these correspond to the adapted entropies for equation (1), see Definition 2.4). We have constructed the numerical scheme (28-30)-(38) in such a way that it preserves solutions of the form (15) for $(c_-, c_+) \in \mathscr{G}^1_\lambda$ (see Proposition 3.1). Regarding the characterization Proposition 2.5[A] of entropy solutions, we also need adapted entropy inequalities with $(c_-, c_+) \in \mathscr{G}^2_\lambda$. While the scheme does not preserve the corresponding entropy solutions (15) exactly (see the numerical results in Section 5), we are able to prove the asymptotic (as $\Delta x \to 0$) preservation property.

Proposition 3.6. *Let v_Δ be the solution of the numerical scheme (26)-(28-30)-(38) with the initial datum $v_0(\cdot) = c(\cdot)$ defined by (15) with $(c_-, c_+) \in \mathscr{G}^2_\lambda$. Then under the CFL condition (40), v_Δ converge to c in $\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ when $\Delta x \to 0$.*

Proof. Discretizing the initial condition $v_0(\cdot) = c(\cdot)$, we have $v_i^0 = c_i$ with

$$c_i = \begin{cases} c_- & \text{if } i \leqslant 0 \\ c_+ & \text{if } i > 0 \end{cases}, \quad (c_-, c_+) \in \mathscr{G}^2_\lambda = [0,\lambda] \times [-\lambda, 0]. \tag{47}$$

By Proposition 39, the numerical scheme exactly preserves the stationary solution in the cases $(c_-, c_+) = (0, -\lambda)$ and $(c_-, c_+) = (\lambda, 0)$. As in the proof of Lemma 3.2, using the monotonicity of H_i to compare v_Δ with this preserved solutions, we find

$$\forall n \geqslant 0, \quad v_i^n \in \begin{cases} [0,\lambda] & \text{if } i \leqslant 0 \\ [-\lambda, 0] & \text{if } i > 0. \end{cases} \tag{48}$$

Note in addition that from the BV_{loc} bounds of Lemmas 3.3, 3.5, we have the L^1_{loc} convergence (up to a subsequence) of the discrete solutions v_Δ, as $\Delta \to 0$, to some limit v. It remains to identify this limit to c (uniqueness of the limit implies the convergence of the whole family v_Δ).

Indeed, we can also consider the numerical scheme on v_Δ as the discretization, with the monotone numerical flux g, of two initial-boundary value problems, the one on $\{x < 0\}$ and the other on $\{x > 0\}$. As far as the problem on $\{x < 0\}$ is concerned, by the definition of g^-_λ the boundary datum in our scheme is given by $(b^n)_n := (v^n_1 + \lambda)_n \subset [0, \lambda]$; and the initial datum is the constant $c_- \in [0, \lambda]$. By the discrete maximum principle, the values of the discrete solution $(v^n_i)_{n>0,i<0}$ lie within $[0, \lambda]$. Thus the limit v of v_Δ is non-negative in $\{x < 0\}$, and, by classical arguments (see e.g. [Vov02]), v is a Kruzhkov entropy solution of the Burgers equation in the half-space in $\{x < 0\}$. Such *non-negative* solution can be interpreted as the unique entropy solution of the Cauchy-Dirichlet problem for the Burgers equation with data $v|_{t=0,x<0} = c_-$ and $v|_{t>0,x=0} = 0$ (the boundary condition is inactive, because the characteristics are outgoing at the boundary). Then we conclude that $v|_{x<0} \equiv c_-$. Analogous arguments permit to find $v|_{x>0} \equiv c_+$. □

As suggested above, we intend to prove that the numerical scheme tends to the entropy solution characterized by [A] (Proposition 2.5). Now we deduce discrete entropy inequalities for $|u - c(x)|$ where c is defined by (15) and $(c_-, c_+) \in \mathscr{G}^1_\lambda \cup \mathscr{G}^2_\lambda$. We have the following "pointwise Kato inequalities".

Proposition 3.7. *Let $(u^n_i)_{i\in\mathbb{Z},n\in\mathbb{N}}$ and $(v^n_i)_{i\in\mathbb{Z},n\in\mathbb{N}}$ be two sequences defined by the numerical scheme (28-30)-(38) and assume that the CFL condition (40) is fulfilled. Then for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$*

$$\frac{|u^{n+1}_i - v^{n+1}_i| - |u^n_i - v^n_i|}{\Delta t} + \frac{G^{n-}_{i+1/2} - G^{n+}_{i-1/2}}{\Delta x} \leqslant 0 \tag{49}$$

where

$$\begin{aligned} \forall i \neq 0 \quad & G^{n-}_{i+1/2} = G^{n+}_{i+1/2} = g(u^n_i \top v^n_i, u^n_{i+1} \top v^n_{i+1}) - g(u^n_i \perp v^n_i, u^n_{i+1} \perp v^n_{i+1}) \\ \textit{and} \quad & G^{n\pm}_{1/2} = g^\pm_\lambda(u^n_0 \top v^n_0, u^n_1 \top v^n_1) - g^\pm_\lambda(u^n_0 \perp v^n_0, u^n_1 \perp v^n_1). \end{aligned}$$

Proof. This proof follows the same guidelines as the proof of classical discrete entropy inequalities (see for instance [GR91]). We use the standard decomposition $|a - b| = a \top b - a \perp b$. First, we have

$$\begin{aligned} u^{n+1}_i \top v^{n+1}_i &= H_i(u^n_{i-1}, u^n_i, u^n_{i+1}) \top H_i(v^n_{i-1}, v^n_i, v^n_{i+1}) \\ &\leqslant H_i(u^n_{i-1} \top v^n_{i-1}, u^n_i \top v^n_i, u^n_{i+1} \top v^n_{i+1}) \end{aligned}$$

by monotonicity (41) of H_i. On the other hand, we have

$$\begin{aligned} u^{n+1}_i \perp v^{n+1}_i &= H_i(u^n_{i-1}, u^n_i, u^n_{i+1}) \perp H_i(v^n_{i-1}, v^n_i, v^n_{i+1}) \\ &\geqslant H_i(u^n_{i-1} \perp v^n_{i-1}, u^n_i \perp v^n_i, u^n_{i+1} \perp v^n_{i+1}) \end{aligned}$$

leading to

$$\begin{aligned} |u^{n+1}_i - v^{n+1}_i| \leqslant\ & H_i(u^n_{i-1} \top v^n_{i-1}, u^n_i \top v^n_i, u^n_{i+1} \top v^n_{i+1}) \\ & - H_i(u^n_{i-1} \perp v^n_{i-1}, u^n_i \perp v^n_i, u^n_{i+1} \perp v^n_{i+1}). \end{aligned}$$

In view of the definitions of H_i and of $G^{n\pm}_{i+1/2}$, this boils down to (49). □

Under the assumption (**H**) (which role is to ensure interface dissipation, compensating for the lack of conservativity), we readily deduce the discrete Kato inequality:

Proposition 3.8. *Let $(u_i^n)_{i,n}$ and $(v_i^n)_{i,n}$ be as in Proposition 3.7. Let $\varphi \in \mathcal{D}([0,T)\times\mathbb{R})$, $\varphi \geqslant 0$, and $\varphi_i^n = \varphi(n\Delta t, i\Delta x)$. Assume the hypothesis (**H**) holds. Then*

$$\Delta t \Delta x \sum_{i\in\mathbb{Z}, n\in\mathbb{N}} |u_i^{n+1} - v_i^{n+1}| \frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} + \Delta x \sum_{i\in\mathbb{Z}} |u_i^0 - v_i^0| \varphi_i^0$$
$$+ \Delta t \Delta x \sum_{i\in\mathbb{Z}^*, n\in\mathbb{N}} G_{i+1/2}^{n\pm} \frac{\varphi_{i+1}^n - \varphi_i^n}{\Delta x} + \Delta t \Delta x \sum_{n\in\mathbb{N}} G_{1/2}^{n+} \frac{\varphi_1^n - \varphi_0^n}{\Delta x} \geqslant 0 \quad (50)$$

Proof. We multiply each inequality (49) by the corresponding value $\Delta t \Delta x \varphi_i^n$, use the summation-by-parts argument and the conservativity of the fluxes for $i \neq 0$. Denoting by $S(\varphi)$ the sum of the first three terms in the left-hand side of (50), we get the inequality

$$S(\varphi) + \Delta t \sum\nolimits_{n\in\mathbb{N}} (G_{1/2}^{n+}\varphi_1^n - G_{1/2}^{n-}\varphi_0^n) \geqslant 0.$$

Add and subtract $G_{1/2}^{n+}\varphi_0^n$ under the sum sign; then we get the last term in (50) plus the sum in n of the terms $(G_{1/2}^{n+} - G_{1/2}^{n-})\varphi_0^n \Delta t$. Assumption (**H0**) (which follows from (**H**)) and the definition of $G_{1/2}^{n\pm}$ ensure that the first factor is non-positive, while φ_0^n is non-negative. Hence (50) follows. □

3.2.3 Convergence

Using the previous results, we easily prove the convergence of the numerical scheme.

Theorem 3.9. *Assume $u_0 \in \mathbf{L}^\infty(\mathbb{R})$. Then, under the CFL condition (40) and assumption (**H**), the numerical scheme (26)-(28-30)-(38) converges in $\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R})$ to the unique entropy solution to problem (1) when Δx tends to 0.*

Proof. In the first step, assume that $u_0 \in \mathrm{BV}(\mathbb{R})$. The BV_{loc} bounds of Lemmas 3.3, 3.5 and the Cantor diagonal extraction argument permit to extract a (not relabelled) subsequence such that $u_\Delta \to u$ in $\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R})$. By Propositions 3.1 and 3.6, for $v_0 = c$ with c given by (15) and $(c_-, c_+) \in \mathscr{G}_\lambda^1 \cup \mathscr{G}_\lambda^2$, the associated solutions v_Δ converge to $v(t,x) := c(x)$.

This allows to pass to the limit in (50). Indeed, the finite difference approximations in (50) of the derivatives φ_t, φ_x of a regular compactly supported test function φ converge in $L^\infty(\mathbb{R}^+\times\mathbb{R})$; and it is well known (see e.g. [Vov02]) that the consistency and Lipschitz continuity of the numerical flux g plus a (local) uniform $BV_{loc}(\mathbb{R})$ estimate on $u_\Delta(t,\cdot)$ allow to conclude that

$$\sum_{i\in\mathbb{Z}^*, n\in\mathbb{N}} G_{i+1/2}^{n\pm} 1\!\!1_{(t^n, t^{n+1}]\times(x_{i-1/2}, x_{i+1/2}]} + \sum_{n\in\mathbb{N}} G_{1/2}^{n+} 1\!\!1_{(t^n, t^{n+1}]\times(x_{-1/2}, x_{1/2}]} \longrightarrow \Phi(u,v)$$

a.e, thus also in $\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R})$ (the last term in the left-hand side is a bit special, but it has no influence at the limit). Then (50) boils down to the family of adapted entropy inequalities (16) corresponding to the case [A] of Proposition 2.5. Thus u is the unique entropy solution of (1) with datum u_0; the accumulation point being unique, the whole family u_Δ converges to u as $\Delta \to 0$.

For the general case $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, we first approximate u_0 by two a.e. convergent sequences of $BV(\mathbb{R})$ functions $(\underline{u}_0^n)_n$ (non-decreasing) and $(\overline{u}_0^n)_n$ (non-increasing) such that $\underline{u}_0^n \leqslant u_0 \leqslant \overline{u}_0^n$. For each $n \in \mathbb{N}$ fixed and all Δt,Δx satisfying the CFL condition, the initial discretization (26) and the monotonicity (41) imply that the associated numerical solutions fulfill

$$\underline{u}_\Delta^n \leqslant u_\Delta \leqslant \overline{u}_\Delta^n, \tag{51}$$

moreover, $(\underline{u}_\Delta^n)_n, (\overline{u}_\Delta^n)_n$ are monotone sequences uniformly bounded in $\mathbf{L}^\infty(\mathbb{R}^+\times\mathbb{R})$. Applying the previous convergence result as $\Delta x \to 0$, we see that the limits $\underline{u}^n, \overline{u}^n$ of $\underline{u}_\Delta^n, \overline{u}_\Delta^n$ are entropy solutions of (1) with data $\underline{u}_0^n, \overline{u}_0^n$, respectively. Then the monotonicity of the sequences and the $\mathbf{L}^1_{loc}$ contraction property (21) in the domain of dependence permits to conclude that $\underline{u}_\Delta^n$ and $\overline{u}_\Delta^n$ both converge to the same limit u in $\mathbf{L}^1_{loc}(\mathbb{R}^+\times\mathbb{R})$. Characterization (16) being stable by $\mathbf{L}^1_{loc}(\mathbb{R}^+\times\mathbb{R})$ convergence of sequences of initial data and entropy solutions, we deduce that u is (the unique) entropy solution of (1) with the datum u_0. From (51) we deduce that u_Δ converge to u, and the proof is complete. □

4 Complements and remarks

4.1 Convergence of the LeRoux regularization

Let us mention a different existence proof, that further justifies the approach of [LST08]. Recall that in [LST08], the authors discussed solutions obtained as limits of equation (11) (or, equivalently, of system (6) with $H = H_\varepsilon$ converging to the Heavyside function), and deduced the possible trace couplings across the interface (see Section 2.1). In this derivation, it was assumed that the solutions to (6) are stationary. Actually, it was deduced that for all $(u_-, u_+) \in \mathscr{G}$, the associated stationary solution $u(t,x) \equiv u_0(x) := u_- 1\!\!1_{\{x<0\}} + u_+ 1\!\!1_{\{x>0\}}$ is the a.e. limit of the (stationary) solutions u^ε to (11) with well-chosen initial data. These are the data given by the profile constructed in Proposition 2.2:

$$u_0^\varepsilon(x) := u_- 1\!\!1_{\{x<-\varepsilon\}} + U_\varepsilon(x) 1\!\!1_{\{|x|\leqslant\varepsilon\}} + u_+ 1\!\!1_{\{x>\varepsilon\}}.$$

Now, we have the following observations. Given H_ε, we have the Kato inequality (10). Consider any L^∞ initial datum v_0 and the corresponding solutions v^ε. We know that the sequence is relatively compact in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$ thanks to the well known precompactness results for the Burgers equation; as usual, we argue along a subsequence. Because $\|u_0^\varepsilon - u_0\|_{L^1}$ vanishes as $\varepsilon \to 0$, u^ε converges to u (both functions are stationary). Then the Kato inequality written for v^ε and u^ε passes to the limit and it yields the adapted entropy inequality (16). We deduce that the limit v of (a subsequence of) v^ε is the unique entropy solution of (1). In this way, using regularization (11), $\varepsilon \to 0$, with fixed initial datum v_0 (or using (6) with fixed v_0 and with H_0 nicely converging to the Heavyside profile), we can pass to the limit and obtain the unique entropy solution of (1) with initial datum v_0.

4.2 Moving sources, multiple sources, problem (2)–(3)

As it was already mentioned in the Introduction, equation $\partial_t u + \partial_x u^2/2 = (V-u)\delta_0(x-Vt)$ reduces to (1) by the simultaneous change of $u-V$, $x-Vt$ into u, x, respectively. Thus it is easy to write the corresponding numerical scheme, provided one allows for non-rectangular space-time cells near the interface (see [ALST10]).

Such a scheme can be also used in the case where several singular sources are present. Intersections of the space-time lines carrying different sources present no difficulty. Indeed, because the solutions are actually continuous in time with values in $L^1_{loc}(\mathbb{R})$ (same arguments as for existence of the interface traces apply), it is sufficient to restart the construction of solutions at the times of intersection.

Further, having in mind the coupled problem (2)–(3), one could consider sources carried by curved particle path $x = h(t)$. The notion of entropy solution (versions [B],[D] of Proposition 2.5) and the uniqueness techniques extend readily to this case; construction of a solution can be done via the change of variables $y = x - h(t)$ or via piecewise affine approximation of $h(\cdot)$. These and further generalizations can be found in [ALST10] and in the future work [ALST].

Such analysis may also apply to the case of singular source

$$\partial_t u(t,x) + \partial_x (u^2/2)(t,x) = -u(t,x)\ \partial_x a(t,x),$$

provided a has the form $a(t,x) = \sum_{i\in\mathbb{N}} a_i(t,x) 1\!\!1_{\Omega_i}(t,x)$ where $(\Omega_i)_i$ is a locally finite partition of $(0,T)\times\mathbb{R}$ with Lipschitz boundaries, the functions a_i are absolutely continuous with respect to x (more precisely, $\partial_x a_i \in \mathbf{L}^1(\Omega_i)$), and the function a satisfies the one-sided Lipschitz condition: there exists $\alpha \in \mathbf{L}^1([0,T])$ such that

$$-\partial_x a(t,x) \leqslant \alpha(t) \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}).$$

Indeed, then the function $-a$ only admits decreasing jumps along Lipschitz curves, where the germ technique of Section 2 applies; and the remaining part of $\partial_x a$ is absolutely continuous and can be treated as a combination of an absorption term and of a Lipschitz source term.

5 Numerical results

First, let us illustrate Proposition (48), *i.e.* the non exact preservation of initial data with $(c_-, c_+) \in \mathscr{G}^2_\lambda$. We use the Rusanov numerical flux, set $\lambda = 1$ and $c_- = c_+ = 0$. Fig. 2 represents the results at $t = 20$ for 10, 100 and 1000 cells. One may see the presence of two numerical boundary layers on both sides of $\{x = 0\}$, which vanish (in $\mathbf{L}^1_{\rm loc}$) when the mesh is refined.

The second test consists in comparing two well-balanced schemes, the first one being based on the Rusanov flux and the second one on the Godunov flux. The initial condition is $u_0 \equiv 1/2$, λ is set to 1 and the final time is 7. For both schemes, the mesh contains 100 cells and the Courant number is 0.4. The results are plotted in Fig. 3. Once again, the Rusanov scheme makes numerical boundary layers appear on both sides from the interface. On the contrary, the Godunov scheme provides exactly the good traces: $\gamma_- u(t) = 1/2$ and $\gamma_+ u(t) = 0$.

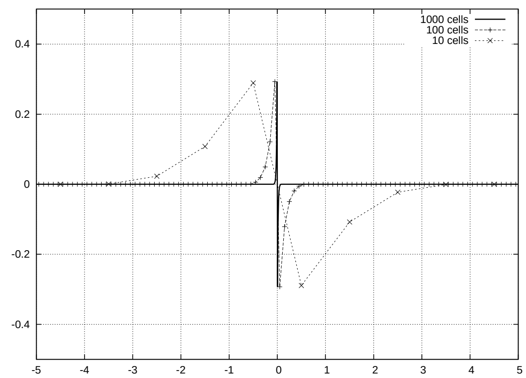

Figure 2: Initial datum with $(c_-, c_+) \in \mathscr{G}^2_\lambda$ for several meshes.

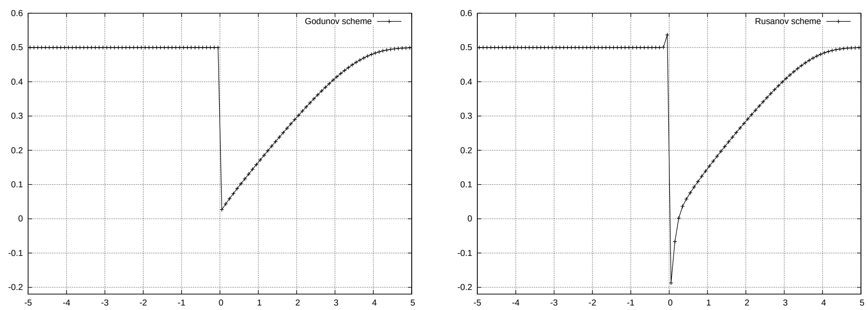

Figure 3: Comparison between well-balanced schemes with the Godunov flux (left) and with the Rusanov flux (right).

A Assumption (H) and numerical fluxes

Lemma A.1. *The Godunov, Rusanov, and Engquist-Osher numerical fluxes verify assumption (***H***) (a fortiori, they verify assumption (***H0***)).*

Proof. Consider the Godunov scheme for $f(u) = u^2/2$. The proof relies on a case-by-case study of monotonicity of

$$\Delta(a, b) = 2(g(a + \lambda, b + \lambda) - g(a, b)).$$

If $a < b$, we have $g(a, b) = \min_{s \in [a,b]} f(s)$. The following situations occur:

Case	$\Delta(a, b)$	$\partial_a \Delta(a, b)$	$\partial_b \Delta(a, b)$
$a < -\lambda, b < -\lambda$	$2b\lambda + \lambda^2$	0	2λ
$a < -\lambda, -\lambda < b < 0$	$-b^2$	0	$-2b$
$a < -\lambda, b > 0$	0	0	0
$-\lambda < a < 0, b < 0$	$(a + \lambda)^2 - b^2$	$2(a + \lambda)$	$-2b$
$-\lambda < a < 0, b > 0$	$(a + \lambda)^2$	$2(a + \lambda)$	0
$a > 0$	$2a\lambda + \lambda^2$	2λ	0

If $a < b$, we have $g(a,b) = \max_{s\in[b,a]} f(s)$. The following situations occur:

Case	$\Delta(a,b)$	$\partial_a\Delta(a,b)$	$\partial_b\Delta(a,b)$
$f(a+\lambda) < f(b+\lambda)$	$2b\lambda+\lambda^2$	0	2λ
$f(a+\lambda) > f(b+\lambda), f(a) < f(b)$	$(a+\lambda)^2 - b^2$	$2(a+\lambda)$	$-2b$
$f(a+\lambda) > f(b+\lambda), f(a) > f(b)$	$2a\lambda+\lambda^2$	2λ	0

In each case, one sees readily that $\partial_a\Delta(a,b)$ and $\partial_b\Delta(a,b)$ are non-negative.

In the Rusanov case, we have $g(a,b) = (a^2+b^2)/4 - (|a|\top|b|)(b-a)/2$, thus

$$\partial_a\Delta(a,b) = (\lambda + |a+\lambda|\top|b+\lambda| - |a|\top|b|)/2 + (a-b)(\mathbb{1}_{|a+\lambda|>|b+\lambda|} - \mathbb{1}_{|a|>|b|})/2.$$

Clearly, the map $t \mapsto |a+t|\top|b+t|$ is 1-Lipschitz, so that the first term on the right-hand side is non-negative. Also the second term is non-negative. Indeed, if, e.g., $a > b$ and $|a| > |b|$, we also have $|a+\lambda| = a+\lambda > |b|+\lambda \geqslant |b+\lambda|$; the other cases are similar. Thus $\partial_a\Delta(a,b) \geqslant 0$; similarly, we get $\partial_a\Delta(a,b) \geqslant 0$.

Finally, in the Engquist-Osher case, $g(a,b) = (a^2+b^2)/4 - \int_a^b |w|/2\,dw$, and $\partial_a\Delta(a,b) = (\lambda + |a+\lambda| - |a|)/2 \geqslant 0$, $\partial_b\Delta(a,b) = (\lambda - |b+\lambda| + |a|)/2 \geqslant 0$. $\square$

Remark 6. Also the flux-splitting flux $g(a,b) = ((a\top 0)^2 + (b\bot 0)^2)/2$ verifies (**H**); but the following modified flux-splitting flux

$$g(a,b) = \frac{(a\top 0)^2 + (b\bot 0)^2}{2} - (b^3 - a^3) \tag{52}$$

does not satisfy assumption (**H0**), while it is monotone and consistent. In spite of the fact that our convergence result does not apply to this case, numerical experiments show convergence properties similar to those of the Rusanov scheme (numerical boundary layers may appear on each side of the interface, see Fig. 4).

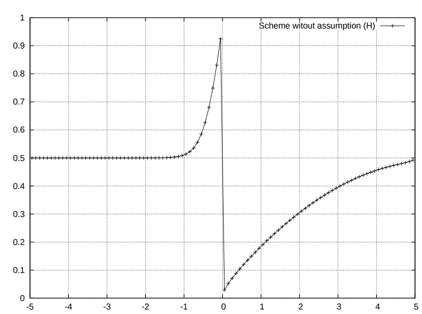

Figure 4: Results with the numerical flux (52)

B The case of a quadratic drag force

We present in this appendix the same kind of results as in the previous sections, for the case of a quadratic source term:

$$\begin{cases} \partial_t u(t,x) + \partial_x(u^2/2)(t,x) = -\lambda\ u(t,x)\ |u(t,x)|\ \delta_0(x), & t>0,\ x\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{53}$$

where $\lambda > 0$. Analysis (of which the details are omitted) follows the same guidelines as for the linear drag force case, with one considerable simplification.

Following the same process of construction, we can describe the germ associated to (53) as follows:

Definition B.1. The *maximal germ* $\mathscr{G}_\lambda \subset \mathbb{R}^2$ associated with (1) is defined as the union $\mathscr{G}_\lambda = \mathscr{G}_\lambda^1 \cup \mathscr{G}_\lambda^3$, where

- $\mathscr{G}_\lambda^1 = \{(a, ae^{-\operatorname{sgn}(a)\lambda}), a \in \mathbb{R}\}$.
- $\mathscr{G}_\lambda^3 = \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^-, -ae^{\lambda} \leqslant b \leqslant -ae^{-\lambda}\}$.

Here, $\mathscr{G}_\lambda^1, \mathscr{G}_\lambda^3$ play the same roles as for the case of the linear drag force; and the part corresponding to $\mathscr{G}_\lambda^2$ can be skipped, which results in a simpler convergence analysis for the scheme. This germ is maximal and L^1-dissipative,

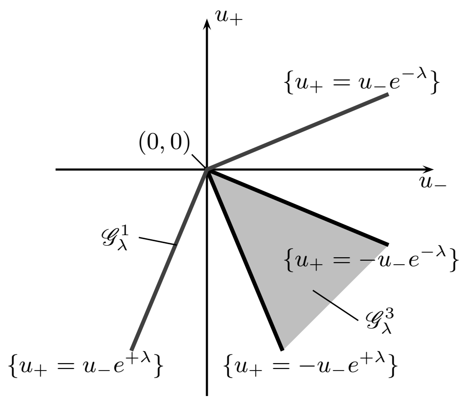

Figure 5: Representation of the admissibility germ $\mathscr{G}_\lambda = \mathscr{G}_\lambda^1 \cup \mathscr{G}_\lambda^3$.

and the part $\mathscr{G}_\lambda^1$ is a definite part of this germ (see [AKR11, AKR10]) in the sense that the following properties hold:

Proposition B.2. *Consider Ξ defined by (8). Then*

i. $\forall (u_-, u_+), (v_-, v_+) \in \mathscr{G}_\lambda, \quad \Xi^{\pm}\big((u_-, u_+), (v_-, v_+)\big) \geqslant 0.$

ii. If a pair $(u_-, u_+) \in \mathbb{R}^2$ verifies:

$$\forall (v_-, v_+) \in \mathscr{G}_\lambda^1 \quad \Xi\big((u_-, u_+), (v_-, v_+)\big) \geqslant 0, \tag{54}$$

then $(u_-, u_+) \in \mathscr{G}_\lambda$.

Then the definitions of entropy solutions and the uniqueness theorem extend quite directly to the case of the quadratic drag force. Further, in the numerical scheme we make the following change into the numerical fluxes: (35) is maintained, and (37) is substituted with

$$\varphi_\lambda^{\pm}(a) = ae^{\mp \operatorname{sgn}(a)\lambda}. \tag{55}$$

By construction, the corresponding scheme is well-balanced with respect to the part $\mathscr{G}_\lambda^1$ of the germ. Thanks to Proposition B.2(ii), this is enough to establish the analogue of the formulation [A] at the limit of the scheme. Thus we have the same well-posedness results for the linear and the quadratic drag forces. Numerically, the main difference of behaviour with the case of a linear drag force is that no boundary layer of the kind seen on Fig. 2 can appear.

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Boris Andreianov
boris.andreianov@univ-fcomte.fr
Laboratoire de Mathématiques CNRS UMR 6623
Université de Franche-Comté
16 route de Gray
25030 Besançon Cedex, France

Nicolas Seguin
nicolas.seguin@upmc.fr
UPMC Université Paris 6 and CNRS UMR 7598
Laboratoire J.-L. Lions
75005 Paris, France